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1

Renormalisation-induced phase transitions for unimodal maps

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July 10, 2008

Abstract

The thermodynamical formalism is studied for renormalisable maps of the interval and the natural potential $-t \log |Df|$. Multiple and indeed infinitely many phase transitions at positive t can occur for some quadratic maps. All unimodal quadratic maps with positive topological entropy exhibit a phase transition in the negative spectrum.

Introduction

This brief note is dedicated to the presentation of a hitherto unstudied phenomenon: phase transitions between positive entropy regimes for the natural potential $-t \log |Df|$ in unimodal dynamics.

In Proposition 9, perhaps the most striking result, we show the existence of infinitely-renormalisable quadratic maps which admit an infinity of phase transitions. In Proposition 4, we show that all interesting quadratic maps exhibit a phase transition in the negative spectrum. In Proposition 8, we show that for each $n \geq 0$, there is a positive measure set of Collet-Eckmann parameters for which the maps admit exactly n phase transitions in a neighbourhood of $[0, 1]$.

In related prior examples of phase transitions (e.g. [1], [2], [7], [8], [12]), all equilibrium states to one side of the phase transition have had zero entropy.

Sarig has interesting results ([16], [18], [17]) on phase transitions in the cadre of countable state Markov shifts; however these are not smooth dynamical systems.

We consider real maps $f : I \rightarrow I$ of a closed interval I and of class C^2 which are unimodal: f has exactly two branches of monotonicity separated by the unique critical point c , and $f(\partial I) \rightarrow \partial I$. The basic family of unimodal maps to consider is the quadratic family: to each positive real parameter a is assigned a quadratic map $f_a : x \rightarrow ax(1 - x)$. For such maps, any f_a -invariant probability measure necessarily lies in the interval $[0, 1]$, and for parameters $a \in (0, 4]$, $f_a : [0, 1] \rightarrow [0, 1]$.

*The author was supported by the EU training network “Conformal Structures and Dynamics”.

We are interested in the natural potential $-t \log |Df|$, where $t \in \mathbb{R}$ is the potential parameter and Df denotes the derivative. The entropy of a measure we denote h_μ and the Lyapunov exponent $\chi_\mu = \int \log |Df| d\mu$. Let $\mathcal{M}(f)$ denote the class of ergodic, f -invariant probability measures. For $\mu \in \mathcal{M}(f)$, we set

$$\mathcal{F}(\mu, t) := h_\mu - t\chi_\mu.$$

This quantity is related to minus the *free energy*. The (*variational*) *pressure* (at parameter t) is then defined as

$$\mathcal{P}(t) := \mathcal{P}_f(t) := \sup_{\mu \in \mathcal{M}(f)} \mathcal{F}(\mu, t) = \sup_{\mu \in \mathcal{M}(f)} (h_\mu - t\chi_\mu).$$

See [14] for a discussion of various definitions of pressure. Provided that the critical orbits are non-periodic, the pressure will be finite. We usually drop from the notation the dependence of the pressure function on the dynamical system $f : I \rightarrow I$ considered. *Equilibrium measures*, or *states*, are measures which realise the supremum. It will also be useful to define, for compact, forward-invariant subsets X of I ,

$$\mathcal{M}(f, X) := \{\mu \in \mathcal{M}(f) : \text{supp}(\mu) \subset X\}$$

and then

$$\mathcal{P}(t, X) := \sup_{\mu \in \mathcal{M}(f, X)} \mathcal{F}(\mu, t).$$

Note that for fixed μ , $\mathcal{F}(\mu, t)$ is linear in t . Convexity of the pressure function follows easily. Moreover, if all measures have non-negative Lyapunov exponents, the pressure function is decreasing. Any measure in $\mathcal{M}(f)$ with negative Lyapunov exponent is supported on an attracting periodic orbit if the critical point of f is non-flat [13]. In particular, in the absence of periodic attractors, Lyapunov exponents of measures are non-negative. A critical point is said to be *non-flat* if there exists a C^2 diffeomorphism $\phi : \mathbb{R} \rightarrow \mathbb{R}$ with $\phi(0) = c$ such that $f \circ \phi$ is a polynomial near the origin.

One expects the pressure function to be smooth, or actually analytic, almost everywhere. We say there is a *phase transition* at t if $\mathcal{P}(t)$ is *not differentiable at t* . This is not the only possible definition. It is weaker than *non-analyticity of $\mathcal{P}(t)$ at t* and differs from the existence of *more than one equilibrium state*. At the phase transitions we exhibit there will be two distinct equilibrium states. Typically, at phase transitions, the equilibrium states will not vary continuously with the potential parameter.

We shall study how renormalisations in unimodal dynamics can lead to phase transitions as the equilibrium states jump between different transitive sets. We exhibit examples of quadratic maps with an infinity of phase transitions at potentials between zero and one, and so where the equilibrium states have positive entropy.

Interesting recent results for thermodynamics of *non-renormalisable maps* of the interval with respect to the natural potential are to be found in the works of Bruin and Todd [2] and of Pesin and Senti [11].

We shall now define renormalisations and explain in a general way how they can give rise to phase transitions before providing details a little later.

A *restrictive interval* J of period $k > 1$ is a closed non-degenerate interval, strictly contained in I and containing the unique critical point c with the following properties. For all i, j with $0 \leq i < j < k$, $f^i(J)$ and $f^j(J)$ overlap in at most one point; $f^k(\partial J) \in \partial J$ and $f^k(J) \subset J$.

The map f is renormalisable of type k if f has a (maximal) restrictive interval J of minimal period k . The unimodal map $\mathcal{R}f := f|_J^k : J \rightarrow J$ is called a *renormalisation* of f of type k . If $k = 2$ the renormalisation is also called a Feigenbaum renormalisation. Since the renormalisation of f is unimodal, its (topological) entropy is bounded by $\log 2$. Thus the entropy of f restricted to $\bigcup_i f^i(J)$ is bounded by $(1/k) \log 2$.

The well known Feigenbaum map is the quadratic map which is infinitely renormalisable of type 2, that is, each renormalisation is of type 2. It has zero topological entropy.

A unimodal map with positive topological entropy $\log s$ is renormalisable of type 2 *if and only if* $\log s \leq (1/2) \log 2$. This can be seen by considering the semi-conjugacy with the piecewise linear *tent map* with slope $\pm s$ — see for example section III.4 of [3], which indeed provides a good background for this paper.

If f is (once-) renormalisable of type $k > 2$ with restrictive interval J of period k , then f has topological entropy strictly greater than $(1/2) \log 2$, as noted before. Denote by X_0 the Cantor set of points which never enter the interior of J under iteration by f . Since the entropy of f restricted to the forward orbit of J is less than or equal to $(1/2) \log 2$, the entropy of f restricted to X_0 is necessarily the entropy of f . Thus all measures of sufficiently large entropy are supported on X_0 .

Since, for t close to zero, the entropy of any equilibrium state must have entropy close to the topological entropy of f , it follows that any such equilibrium state must be supported on X_0 . Supposing that X_0 is hyperbolic, existence of such equilibrium states follows from the general theory for hyperbolic maps ([15]). On the other hand, for sufficiently large $t > 0$, one can expect any equilibrium measure to give positive measure to (the interior of) J .

We now explain why X_0 often is hyperbolic. The Mañé hyperbolicity theorem [9], states that if f is of class C^2 , any forward invariant compact set disjoint from critical points and non-repelling periodic points is a hyperbolic set. If f has negative Schwarzian derivative, then any parabolic periodic point is attracting on one side and contains a (the) critical point in its immediate basin of attraction. In particular, if ∂J does not contain a parabolic periodic point, then X_0 contains no parabolic periodic points, so X_0 is hyperbolic.

We would like to insist that the dynamics of f restricted to X_0 and of $\mathcal{R}f : J \rightarrow J$ are essentially unconnected. If one no longer requires f to be analytic then one can perturb the maps essentially independently. For this reason one can expect that (even for quadratic maps) a renormalisation of type > 2 will lead to a phase transition. This is perhaps the key idea of the paper.

It follows that with strings of renormalisations one can find unimodal maps

whose pressure functions admit various different behaviours: linear followed by strictly convex; strictly convex followed by linear followed by strictly convex followed by linear; piecewise linear etc. Constructing such maps is left as an exercise for the curious reader.

One could ask whether similar phenomena to those exhibited in this paper can occur for rational maps. They cannot: in [4] we show that there exists at most one equilibrium state with positive entropy for each t . This is essentially due to the *eventually onto* property in rational dynamics.

Results

We now proceed to formally state some results and outline their proofs, starting with some simple observations.

Definition 1 *Let $l_1 \leq l_2 \in \mathbb{R} \cup \{\pm\infty\}$. We say a function $h : X \rightarrow \mathbb{R}$, where $X \subset \mathbb{R}$, has (l_1, l_2) -bounded slope if, for all $x < y \in X$,*

$$l_1(y - x) \leq h(y) - h(x) \leq l_2(y - x).$$

Lemma 2 *Let $h, h' : \mathbb{R} \rightarrow \mathbb{R}$ be two real functions. Suppose h has (l_1, l_2) -bounded slope and h' has (m_1, m_2) -bounded slope, where $l_1 \leq l_2 < m_1 \leq m_2$. Then there exists a unique $x_0 \in \mathbb{R}$ for which $h(x_0) = h'(x_0)$. Moreover, the function $\max(h, h')$ is not differentiable at x_0 , and it coincides with h on $\{x \leq x_0\}$ and with h' on $\{x \geq x_0\}$.*

Proof: Evident. □

For $f : I \rightarrow I$ be a map of the interval and X a compact, forward-invariant subset of I , let $\overline{\chi}(X)$ denote the supremum of the Lyapunov exponents of measures in $\mathcal{M}(f, X)$ and $\underline{\chi}(X)$ the infimum.

Lemma 3 *The restricted pressure function $P(t, X)$ has $(-\overline{\chi}(X), -\underline{\chi}(X))$ -bounded slope.*

Proof: This follows easily from the definitions. □

Define \mathcal{Q} as the set of $a \in [3, 4]$ for which the critical point is not periodic. Note that all *interesting* quadratic maps have parameters in $[3, 4]$: if a is greater than 4 then f restricted to the non-wandering set is hyperbolic and conjugate to the full shift on two symbols; for $a \in [0, 3]$ there are only a finite number of periodic orbits.

For $a \in [3, 4] \setminus \mathcal{Q}$, the pressure, as defined, is infinite for all strictly positive t . One can alternatively restrict the definition of pressure to the supremum over measures living on the Julia set, and thus avoid this problem, since the Lyapunov exponent of a measure being negative implies the measure sits on a periodic attractor ([13]). The following proposition could then go, *for all quadratic unimodal maps f with positive topological entropy, there is a phase transition...* In any case, for $a \notin \mathcal{Q}$, the reader can formulate corresponding statements without too much difficulty.

Proposition 4 *For all $a \in \mathcal{Q}$, the pressure function of the quadratic map f_a , with topological entropy denoted h_a , admits a phase transition at some $t \leq -h_a/(\log a - h_a)$.*

Proof: The phase transition will be caused by the unit mass μ_0 sitting on the repelling fixed point at zero, whose Lyapunov exponent is $\log a$. Thus $\mathcal{P}(t, \{0\}) = -t \log a$ and in particular $\mathcal{P}(t, \{0\})$ has $(-\log a, -\log a)$ -bounded slope.

If $a = 4$, we are dealing with the Chebyshev map. This map is smoothly conjugate on the interior of $[0, 1]$ to the full tent map with slope ± 2 , so all measures other than μ_0 have Lyapunov exponent equal to $\log 2$. It is straightforward to verify that there is a unique phase transition at $t = -1$.

Now suppose $a \neq 4$. Any other measure in $\mathcal{M}(f_a)$ lives on $X := [f^2(c), f(c)]$ where the norm of the derivative is bounded away from (and below) a . Thus $\overline{\chi}(X) < \log a$, and $\underline{\chi}(X) > -\infty$ since $a \in \mathcal{Q}$. We know $\mathcal{P}(t, X)$ has $(\overline{\chi}(X), \underline{\chi}(X))$ -bounded slope.

We have $\mathcal{P}(t) = \max(\mathcal{P}(t, \{0\}), \mathcal{P}(t, X))$. Applying Lemma 3 will give a required phase transition at some parameter t_0 . To see that $t_0 \leq -h_a/(\log a - h_a)$, note first that for each a , $h_a = \mathcal{P}(0) = \mathcal{P}(0, X)$, and that $0 = \mathcal{P}(0, \{0\})$. Thus if $h_a = 0$ then $t_0 = 0$, by uniqueness of t_0 .

If $h_a > 0$ let μ_a be the (unique) measure of maximal entropy. The graphs of $\mathcal{F}(t, \mu_0)$ and $\mathcal{F}(t, \mu_a)$ intersect when $t = -h_a/(4 - \chi_{\mu_a})$. The result follows upon applying Ruelle's inequality: $h_\mu \leq \max(0, \chi_\mu)$. \square

Remark: We shall show later that there are unimodal maps without negative Schwarzian derivative but with non-flat critical point which admit phase transitions in the negative spectrum (i.e. at some negative t) where the equilibrium states on both sides of the phase transition have positive entropy.

Remark: Pesin has conjectured that there should be a positive measure set of parameters in the neighbourhood of each Misiurewicz parameter for which, for each real t , there exists a unique equilibrium measure, and such that the pressure function is real analytic everywhere. The proposition above shows that one should at least restrict one's attention to measures sitting on $[f^2(c), f(c)]$ in the non-renormalisable setting. Under this additional hypothesis, we suspect the conjecture to be true. Note that near any non-renormalisable Misiurewicz parameter will be positive measure sets of both non-renormalisable and renormalisable Collet-Eckmann parameters. Indeed, such maps are accumulated by both non-renormalisable and renormalisable post-critically finite maps, around which one can apply the Benedicks-Carleson construction to find the required positive measure sets ([21], [20]).

Definition 5 *We shall call a collection of maps $\{g_\alpha\}_{\alpha \in \Delta}$ a full unimodal family provided:*

- Δ is an interval;
- each $g_\alpha : I_\alpha \rightarrow I_\alpha$ is a C^3 unimodal map of the interval I_α and has negative Schwarzian derivative and non-flat critical point;

- the boundary of I_α depends continuously on α ;
- rescale g_α by an affine, orientation-preserving conjugacy to get a unimodal map $g_\alpha^* : [0, 1] \rightarrow [0, 1]$; then $\alpha \mapsto g_\alpha^*$ is continuous for the C^1 topology on $\{g_\alpha^*\}_{\alpha \in \Delta}$;
- for every $a \in (0, 4]$ there is an $\alpha \in \Delta$ and a conjugacy between g_α and the quadratic map f_a .

Of course, the quadratic family $\{f_a\}_{a \in (0, 4]}$ is a full unimodal family. Requiring negative Schwarzian derivative and non-flat critical point means that each periodic attractor of g_α is essential (its immediate basin contains the critical point) and that wandering intervals do not exist.

Let $\{g_\alpha\}_{\alpha \in \Delta}$ be a full unimodal family. Let $\alpha_0 \in \Delta$ be such that g_{α_0} is conjugate to the Chebyshev map f_4 , but so that there are α arbitrarily close to α_0 which are not conjugate to f_4 . Then it follows from the definitions and continuity that there exists a sequence $\{\alpha_k\}_{k \geq 2}$ with $\lim_{k \rightarrow \infty} \alpha_k = \alpha_0$ such that g_{α_k} has critical orbit satisfying $g_{\alpha_k}^k(c) = c$ and either

$$g_{\alpha_k}^i(c) < c < g_{\alpha_k}(c)$$

or

$$g_{\alpha_k}^i(c) > c > g_{\alpha_k}(c)$$

for all i with $2 \leq i < k$. The first possibility holds for maps with the same orientation as the quadratic family. Note that c is contained in a restrictive interval of period n . For each $k \geq 2$, let Δ_k denote the connected component containing α_k of parameters $\alpha \in \Delta$ such that g_α is renormalisable of type k . Denote by J_α the corresponding maximal restrictive intervals. If for each $k \geq 2$, $\{g_\alpha^k : J_\alpha \rightarrow J_\alpha\}_{\alpha \in \Delta_k}$ is a full unimodal family then we shall call $\{\alpha_k, \Delta_k\}_{k \geq 2}$ a *suitable sequence*.

The following lemma follows from Sections II.4-5 of [3] (see in particular pages 148-149) which contain stronger results and details with less restrictive definitions. We only need to remark that one can choose α_k converging to α_0 because the same holds for the quadratic family.

Lemma 6 *Given a full unimodal family $\{g_\alpha\}_{\alpha \in \Delta}$, there exists $\alpha_0 \in \Delta$ with g_{α_0} conjugate to f_4 and a suitable sequence $\{\alpha_k, \Delta_k\}_{k \geq 2}$ with $\lim_{k \rightarrow \infty} \alpha_k = \alpha_0$.*

Given any full unimodal family we fix a suitable sequence $\{\alpha_k, \Delta_k\}_{k \geq 2}$ and call maps g_α with $\alpha \in \Delta_k$ *simply renormalisable of type k* (these are not the only renormalisable maps of a given type k).

For $\alpha \in \Delta_k$, denote by X_α the set of points which never enter the interior of the restrictive interval J_α . Let us call a unimodal map $g : J \rightarrow J$ *trivial* if g only has one fixed point. In this case the orbit of every point converges to the boundary fixed point.

Lemma 7 *Let $\{\alpha_k\}_{k \geq 2}$ be a suitable sequence for a full unimodal family $\{g_\alpha\}_{\alpha \in \Delta}$. Given any $\varepsilon > 0$, there exists $K > 1$ such that, for each $k \geq K$ and all $\alpha \in \Delta_k$,*

$$1 - \varepsilon < \text{HD}(X_\alpha).$$

Moreover, if $\mathcal{R}g_\alpha$ is not trivial, then $\text{HD}(X_\alpha) < 1$.

Proof: Let $\alpha_0 := \lim_{k \rightarrow \infty} \alpha_k$. The map g_{α_0} is topologically conjugate to the Chebyshev map f_4 . It has hyperbolic dimension equal to 1 (see e.g. [5]). Therefore, denoting by c the critical point of g_{α_0} , there is a $\delta > 0$ such that the set of points never entering $B(c, \delta)$ is a compact hyperbolic set of dimension greater than $1 - \varepsilon/2$.

This set persists under perturbations, so for all α sufficiently close to α_0 , the set of points never entering $B(c, \delta/2)$ say, under iteration by g_α , is a compact hyperbolic set of dimension greater than $1 - \varepsilon$. Taking $\alpha \in \Delta_k$ for k large implies α is close to α_0 , by hypothesis. We now explain why the restrictive interval J_α is contained in $B(c, \delta/2)$ for large k . Suppose not, then there exists a $\nu > 0$ such that for k large and $\alpha \in \Delta_k$, J_α compactly contains $W := B(c, \nu)$. Some iterate of g_{α_0} maps W onto I_{α_0} . Thus for all α sufficiently close to α_0 , some iterate of g_α maps W onto an interval compactly containing J , contradiction. The first statement follows.

Now, since g_α has negative Schwarzian derivative, each non-repelling periodic orbit contains a critical point in its immediate basin of attraction. Thus if ∂J_α contains a parabolic periodic point, then $\mathcal{R}g_\alpha$ is trivial. Otherwise, X_α does not contain any non-repelling periodic points. Then X_α is hyperbolic by, for example, the Mañé hyperbolicity theorem, and so has Hausdorff dimension strictly less than one. \square

Proposition 8 *For each $n \geq 0$, there exists a positive measure set A of parameters such that, for all $a \in A$,*

- *the pressure function $\mathcal{P}_{f_a}(t)$ admits exactly n phase transitions in $(0, 1)$ and no additional phase transitions on neighbourhoods of 0 and of 1;*
- *the pressure function is piecewise analytic on a neighbourhood of $[0, 1]$ and is strictly decreasing;*
- *f_a admits an absolutely continuous invariant probability measure.*

Proposition 9 *There exist uncountably many parameters a for which the pressure function of the quadratic map f_a admits:*

- *(countably) infinitely many phase transitions in $(0, 1)$;*
- *exactly one phase transition at some $t_a < 0$;*
- *no phase transition at $t = 0$ nor for $t \geq 1$.*

Between the phase transitions the pressure function is analytic. For $t < 1$ it is strictly decreasing and for $t \geq 1$, $\mathcal{P}(t) = 0$. At each phase transition there exist exactly two distinct equilibrium states.

Proof of Propositions 8, 9: Consider sequences of integers $(a_1 b_1 a_2 b_2 \dots a_n b_n)$ where $a_i \geq 3$ and $b_i \geq 0$. Denote by \mathcal{A}_1 the set of parameters $a \in [3, 4]$ such that f_a is simply renormalisable of type a_1 . Denote by $\mathcal{B}_1 \subset \mathcal{A}_1$ the set of parameters in \mathcal{A}_1 such that $\mathcal{R}f_a$ is b_1 -times renormalisable of type 2. Define inductively k_n , \mathcal{A}_n and \mathcal{B}_n for $n > 1$ as follows:

- $k_1 := 0$; $k_n := k_{n-1} + 1 + b_{n-1}$;
- there is a subset Δ of \mathcal{B}_n such that $\{\mathcal{R}^{k_n} f_a\}_{a \in \Delta}$ is a full unimodal family by Lemma 6; fix a suitable sequence and let \mathcal{A}_{n+1} be the set of parameters $a \in \mathcal{B}_n$ such that $\mathcal{R}^{k_n} f_a$ is simply renormalisable of type a_{n+1} and $\mathcal{R}^{k_n+1} f_a$ is not trivial;
- \mathcal{B}_{n+1} is the set of parameters $a \in \mathcal{A}_{n+1}$ such that $\mathcal{R}^{k_n+1} f_a$ is b_{n+1} -times renormalisable of type 2.

For $n \geq 1$ set J_n^a as the restrictive interval of $\mathcal{R}^{k_n} f_a$. For $n = 1$ set $L_1^a := I$ and for $n \geq 2$ define L_n^a as the restrictive interval of $\mathcal{R}^{k_n-1} f_a$ (so L_n^a is the domain of $\mathcal{R}^{k_n} f_a$). Then define X_n^a as the set of points in L_n^a which never get mapped into the interior of J_n^a under iteration by $\mathcal{R}^{k_n-1} f_a$.

Given any set K , we shall write $\mathcal{O}_{f_a}(K) := \bigcup_{i \geq 0} f_a^i(K)$ for the smallest, forward-invariant set for f_a containing K .

We shall use the following inductive step. If we fix any sequence of integers $(a_1 b_1 \dots b_{n-1} a_n)$ with $a_i \geq 3$ and $b_i \geq 0$ then provided b_n and a_{n+1} are sufficiently large, the following properties hold.

1.

$$\overline{\chi}(\mathcal{O}_{f_a}(L_n^a)) < (1/2) \inf_{i < n} \underline{\chi}(X_i^a);$$

2. for all $i < n$

$$0 < 1 - \text{HD}(X_n^a) < \text{HD}(X_n^a) - \text{HD}(X_i^a). \quad (1)$$

The first point follows on taking $b_n \geq 1$ sufficiently large. Indeed, according to the renormalisation theory of Sullivan ([19]; [3], chapter VI), for $a \in \mathcal{A}_n$, the derivative of $\mathcal{R}^{k_n+m} f_a$ is uniformly bounded in m and in a . Thus the Lyapunov exponents of any f_a -invariant measures on $\mathcal{O}_{f_a}(L_n^a)$ are exponentially small in b_n .

To show the second point, we remark that, having fixed some $b_n \geq 1$, for all $a \in \mathcal{B}_n$ there is an $\varepsilon > 0$ such that $\text{HD}(X_i^a) < 1 - \varepsilon$ for all $i < n$, by a uniform hyperbolicity argument. Then Lemma 7 gives the required a_{n+1} .

Thus, for each $n > 0$ there are plenty of choices for a sequence $(a_1 b_1 \dots a_n b_n)$ such that we have a corresponding parameter interval \mathcal{A}_n and the sets X_i^a verify

$$\log |Df(0)| > \overline{\chi}(\mathcal{O}_{f_a}(X_i^a)) \geq \underline{\chi}(\mathcal{O}_{f_a}(X_i^a)) > (1/2) \overline{\chi}(\mathcal{O}_{f_a}(X_{i+1}^a)) \quad (2)$$

and

$$0 < 1 - \text{HD}(X_{i+1}^a) < \text{HD}(X_{i+1}^a) - \text{HD}(X_i^a). \quad (3)$$

for $i = 1, \dots, n-1$ and all $a \in \mathcal{A}_n$.

Now apply Lemma 2 to the restricted pressure functions $\mathcal{P}(t, \mathcal{O}_{f_a}(X_i^a))$ and $\mathcal{P}(t, \mathcal{O}_{f_a}(X_{i+1}^a))$. The slope inequalities (2) give a unique intersection at some $t =: t_i$. The dimension estimates (3), together with the slope estimates (2), imply by elementary geometry that

$$t_i < \text{HD}(\mathcal{O}_{f_a}(X_i^a)) < t_{i+1}$$

for each i , and that $\mathcal{P}(t_i) = \mathcal{P}(t_i, \mathcal{O}_{f_a}(X_i^a)) > 0$. Again using (2), and using analyticity of the restricted pressure functions, there is a phase transition at each t_i , $i = 1, \dots, n$, and these are the only phase transitions on $(-\varepsilon, t_n)$ for some $\varepsilon > 0$.

Let us show Proposition 8. Following the work of Pesin and Senti [11], there exists a Lebesgue positive measure subset A of Collet-Eckmann parameters $a \in \mathcal{A}_n$ for which a unique equilibrium state exists for each t in a neighbourhood of $[0, 1]$ for $\mathcal{R}^{k_n} f_a$. Moreover, using the techniques of Bruin and Todd [2] the pressure functions for the renormalised maps for these parameters can be shown to be analytic on a neighbourhood of $[0, 1]$. Thus for these parameters we also have analyticity on $(t_n, 1 + \varepsilon)$ for some $\varepsilon > 0$. Since $a \in A$ are Collet-Eckmann parameters, $\underline{\chi}(I) > 0$ so the pressure function is strictly decreasing, and f_a admits an absolutely continuous invariant probability measure.

For Proposition 9, it suffices for us to consider infinite sequences of the form $(a_1 b_1 a_2 b_2 \dots)$ satisfying (3) and (2) for all truncations, and the corresponding parameters $\bigcap_n \mathcal{A}_n$ (for each sequence this infinite intersection contains exactly one parameter). We remark that f restricted to $\omega(c)$, the omega-limit set of the critical point, is uniquely ergodic and its measure μ has zero entropy. The Lyapunov exponent of μ is non-negative by [13]. One can use [6] to show it is not strictly positive; otherwise μ -almost every point would be contained in arbitrarily small (restrictive) intervals getting mapped by an iterate of f onto some fixed interval, contradiction. Thus $\mathcal{F}(\mu, t) = 0$ for all t . Thus $\mathcal{P}(t) = 0$ for $t \geq 1$. \square

Proposition 10 *There exists a smooth unimodal map with non-flat critical point for which the pressure function admits phase transitions at s and t for some $s < 0 < t$. The pressure function $\mathcal{P}(t)$ is strictly convex and analytic on each of $(-\infty, s)$, (s, t) , $(t, +\infty)$.*

Proof: Let f_n be a quadratic map which is renormalisable of type 3, whose renormalisation is $n-1$ times renormalisable of type 2 for some $n \geq 2$, and whose final renormalisation, $\mathcal{R}^n f_n$ say, is topologically conjugate to the Chebyshev map $x \mapsto 4x(1-x)$. Let J' denote the domain of $\mathcal{R}^n f_n$.

Let X_n denote the largest transitive hyperbolic compact set of points which never enter the interior of the first renormalisation interval under iteration by f_n . Then X_n does not contain $\{0\}$. Standard considerations give $H > 0$ and $\lambda > 0$ such that for all $n \geq 2$, $H < \text{HD}(X_n)$ and $\lambda < \underline{\chi}(X_n) < \overline{\chi}(X_n) < \log 4$. The restricted pressure function $\mathcal{P}(t, X_n)$ is analytic, cuts the t -axis at $\text{HD}(X_n) < H < 1$ and has $(-\log 4, -\lambda)$ -bounded slope. We write f for f_n and X for X_n in what follows, dropping the dependence on n .

We want to modify f . Since f is a quadratic Misiurewicz map, we have $\underline{\chi}([0, 1]) > 0$. Denote by β the point in $\partial J'$ fixed by $\mathcal{R}^n f$, and by α the other, internal, fixed point of $\mathcal{R}^n f$. Let the open interval V , $\alpha \in V \subset J'$ verify the following:

- $f^k(\partial V) \cap V = \emptyset$ for all $k \geq 0$;
- $|D\phi_V| \geq 1 + \underline{\chi}([0, 1])/2$, where ϕ_V denotes the first return map to V .

Note that if h is some smooth function and $c_1, c_2 \in \mathbb{R}$, we write $c_1 < |Dh| < c_2$ if $c_1 < |Dh(x)| < c_2$ for all x in the domain of h .

The first point ensures that each branch of the first return map to V will map its domain diffeomorphically onto V (cf. the *nice intervals* of Martens [10]). Indeed, let A and B be connected components of $f^{-k}(V)$ and $f^{-l}(V)$ respectively for some k, l with $0 \leq k < l$ and suppose $\partial A \cap B \neq \emptyset$, so $f^l(\partial A) \cap V \neq \emptyset$. But then $f^{l-k}(\partial V) \cap V \neq \emptyset$, contradiction. This implies that the return time is constant on each connected component of the domain of the first return map.

To find arbitrarily small V satisfying the first point one can use density of periodic orbits. For the second, use negative Schwarzian derivative and extendibility of branches onto J' . By the Koebe Principle ([3], Theorem IV.1.2), taking V small enough, on each branch of ϕ_V the derivative is approximately constant. The lower bound on Lyapunov exponents gives a lower bound for the derivative at the fixed point of each branch (a periodic point of f).

Let $\gamma \neq \alpha$ be another fixed point of ϕ_V and modify f (see figure 1) on neighbourhoods of α and γ compactly contained in their respective branch domains (of ϕ_V), to get a C^∞ topologically conjugate map g with first return map ψ_V to V so that

1. $|D\psi_V(\alpha)| > 2^{2^n 12}$;
2. $1 < |D\psi_V(\gamma)| < 1 + \underline{\chi}_f([0, 1])/2$;
3. $1 + \underline{\chi}_f([0, 1])/3 \leq |D\psi_V|$.

The first point implies that the measure sitting on the orbit of α (with period $2^{n-1}3$) has Lyapunov exponent greater than $4 \log 4$. The second point implies that the measure sitting on the orbit of β cannot be an equilibrium state (see [8] for a discussion of how this can cause a problem in the rational context). The third point means that we have not created any parabolic or attracting orbits.

Let J denote the set $\bigcup_{i \geq 0} g^i(J')$. The restricted pressure function $\mathcal{P}_g(t, J)$ is analytic. Indeed the first return map to the interval, delimited by the two preimages of α under $\mathcal{R}^n g$, is a hyperbolic induced map and the usual techniques for it (e.g. [2]) can be applied. All measures other than the one sitting on the orbit of β *lift* to this induced map.

We have $\mathcal{P}(0, J) = (\log 2)/(2^{n-1}3)$, $\mathcal{P}(1, J) = 0$. Because of convexity, for all n large enough that $\mathcal{P}(0, J) < H\lambda$, this pressure function has a transverse (and unique in $(0, 1)$) intersection with $\mathcal{P}(t, X)$ at some $t \in (0, 1)$, see figure 2.

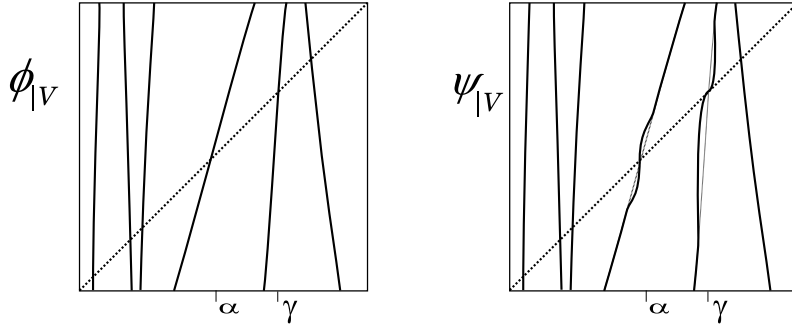


Figure 1: Modifying f modifies the first return map.

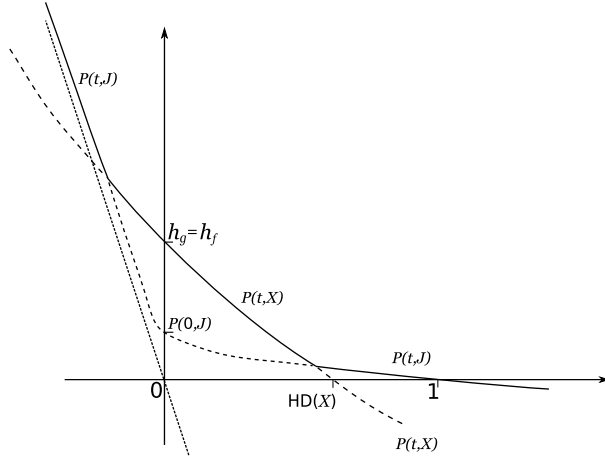


Figure 2: Graphs of the restricted pressure functions.

Now we need to see what happens in the negative spectrum. $\mathcal{P}(t, J) \geq -t4 \log 4$ due to the measure sitting on the orbit of α . The entropy $h_g = h_f$ of g is greater than $(\log 2)/2$ (which is at least 2^{n-1} times $\mathcal{P}(0, J)$). One can deduce that there is an intersection of the graphs of the restricted pressure functions at some maximal $t =: s$ satisfying $-h_g/(3 \log 4) < s < 0$. Then a simple calculation again gives that the derivative $|D\mathcal{P}(s, J)| \geq (h_g - \mathcal{P}(0, J))/|s|$. Since $n \geq 2$,

$$|D\mathcal{P}(s, J)| > (h_g/(2|s|)) > \log 4 > |D\mathcal{P}(s, X)|$$

so, by convexity, this is the only intersection in the negative spectrum. \square

We finish with a remark which first arose in conversation with Juan Rivera-Letelier. Let f be the analytic map with quadratic fixed point, renormalisable of type $k > 2$, which is a fixed point of the renormalisation operator (*i.e.* $\mathcal{R}f$ is an affine rescaling of f). Then there are exactly two phase transitions of the pressure function, one at some negative $t < 0$, the other at some $t_* > 0$ equal

to the dimension of the hyperbolic set of points which never enter the interior of the restrictive interval. At t_* there are an *infinity* of equilibrium states, one on each level of the *filtration* into transitive hyperbolic sets, and one on the omega-limit set of the critical point.

Acknowledgements

We would like to thank the referees and J. Rivera-Letelier and M. Todd for their helpful and interesting comments, questions and suggestions.

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